UDC 62-50

ON A CLASS OF DIFFERENTIAL GAMES OF PURSUIT

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A differential two person game with fixed time of termination is considered. The dynamics of the pursued object is described by a nonlinear differential vector equation, and the pursuing object executes a simple motion; its vector diagram consists of a polyhedron of a special form. The class of these polyhedrons includes e.g. simplexes. Additional assumptions are used to show that the set of all points from which the pursuer may realize the arrival of the phase vector at the terminal set exactly at the instant τ, coincides with the analogous set for the programed strategies. An example is given.

In the theory of differential games, the problem of determining in which case the capture may be realized by the time of the first absorption, is of considerable importance [1-3]. A class of games for which this condition holds, is described below.

Let us consider a differential game described by the following equations of motion:

$$y^{\cdot} = f(y, v), \quad z^{\cdot} = u \tag{1}$$

$$y \in E^{\mu}, \; z \in E^{\nu}, \; u(t) \in P \subset E^{\nu}, \; v(t) \in Q \subset E^{\varkappa}$$

Here $u(t)$ and $v(t)$ are the controls of the pursuing and pursued objects, E^{μ}, E^{ν}, and $E^{\varkappa}$ denote the Euclidean spaces of dimension μ, ν and $\varkappa$ respectively. The terminal set is given in the form

$$M = \{x = (y, z) \in E^{\mu} \times E^{\nu} \mid z = \varphi(y), \; y \in G \subset E^{\mu}\} \tag{2}$$

where φ is a continuous function mapping E^{μ} in E^{ν} and the product $E^{\mu} \times E^{\nu}$ is an Euclidean space of dimension $\mu + \nu$).

We introduce the following constraints: the function $f(y, v)$ is continuous in y and v, satisfies the Lipschitz condition in y and a constant C exists such that the inequality $\mu + \nu$.

$$(y \cdot f(y, v)) \leqslant C(\|y\|^2 + 1) \tag{3}$$

holds for all y and v. The functions $u(t)$ and $v(t)$ are measurable, the set G is closed and the sets P and Q are compacts. In addition the set P is convex and for every $0 < \varepsilon < 1$ and for vector $z \in E^{\nu}$ there exists a vector $r \in P$ such that

$$P \cap ((1 - \varepsilon) P + z) \subset (1 - \varepsilon) P + \varepsilon r \tag{4}$$

Let $D(y_0, \tau)$ be a set of attainability, for the controlled process described by the first equation of (1), from the point y_0 by the time τ, i.e. if we denote by Y the set of solutions of this equation with the initial condition $y(0) = y_0$, then

$$D(y_0, \tau) = \{y_1 \in E^{\mu} \mid \exists y(t) \in Y, \; y_1 = y(\tau_1)\}$$

We assume that the function φ is such that for every $y \in E^{\mu}$ the set $\varphi(D(y, \tau))$

which traverses the vector $\varphi(y_1)$ when the vector y_1 traverses $D(y, \tau)$, satisfies the inclusion

$$\varphi(D(y, \tau)) \subset \varphi(y) + \tau P$$

The above condition guarantees that, if in the initial position (when $t = 0$) $z(t) = \varphi(y(t))$, then the pursuer, in the course of his motion, knowing the control of the pursued player, can maintain this equality.

Let us define the set $T_\tau(M)$ consisting of such, and only such points $x_0 = (y_0, z_0)$, that for every measurable control $v(t)$ there exists a measurable control $u(t)$ such that the solution of the system (1) corresponding to the initial x_0 $x(\tau) \in M$.

Taking into account the special form of the second equation of motion (1) and of the terminal set (2), we can write

$$T_\tau(M) = \{(y_0, z_0) \in E^\mu \times E^\nu \mid D(y_0, \tau) \subset G, \varphi(D(y_0, \tau)) \in z_0 + \tau P\}$$

Let $\omega = \{0 = t_0 < t_1 < \ldots < t_k = \tau\}$ be an arbitrary final subdivision of the segment We assume that $\tau_i = t_i - t_{i-1}$, $1 \leqslant i \leqslant k$ and

$$T_\tau^\circ(M) = \bigcap_\omega T_{\tau_1}(T_{\tau_2}(\ldots(T_{\tau_k}(M))\ldots)).$$

The meaning of the set $T_\tau^\circ(M)$ is, that in the differential game of pursuit (i.e. with discrimination of the pursued), the set separates out those, and only those initial points from which the pursuer can realize the arrival of the phase vector at the terminal set M exactly at the instant τ [3.4].

Theorem. If the assumptions (1) - (5) all hold, then

$$T_\tau^\circ(M) = T_\tau(M), \quad \forall \tau \geqslant 0 \tag{6}$$

Proof. We shall show that the following inclusion takes place:

$$T_{\tau_1}(T_{\tau_2}(M)) \supset T_{\tau_1+\tau_2}(M), \quad \forall \tau_1, \tau_2 > 0 \tag{7}$$

Let the point $x_0 = (y_0, z_0) \in T_{\tau_1+\tau_2}(M)$ and assume that the pursued player has chosen a control $v(t)$ on the segment $[0, \tau_1]$. Denoting the solution of the first equation of (1) with initial condition $y(0) = y_0$ by $y(t)$ and setting $y_1 = y(\tau_1)$, we have

$$\varphi(D(y_1, \tau_2)) \subset \varphi(D(y_0, \tau_1 + \tau_2)) \subset z_0 + (\tau_1 + \tau_2) P \tag{8}$$

Condition (5) yields

$$\varphi(D(y_1, \tau_2)) \subset \varphi(y_1) + \tau_2 P \tag{9}$$

Taking into account (4), we conclude from (8) and (9) that a vector $r \in \tau_1 P + z_0$, exists such that $\varphi(D(y_1, \tau_2)) \subset \tau_2 P + r$. Choosing $u(t) = (r - z_0) / \tau_1$ with $t \in [0, \tau_1]$ and denoting by $z(t)$ the solution of the second equation of (1) with initial condition $z(0) = z_0$, we find that $z(\tau_1) = r$ and consequently $(y(\tau_1), z(\tau_1)) \in T_{\tau_2}(M)$, i.e. $x_0 \in T_{\tau_1}(T_{\tau_2}(M))$. Thus the inclusion (7) holds and from this follows (6), Q. E. D.

Now we shall give a number of assertions, without proof, concretizing the condition (4). We use the terminology given in [5].

1°. All simplexes satisfy the condition (4).

2°. If the polyhedrons $P_1 \subset E^{\nu_1}$ and $P_2 \subset E^{\nu_2}$ satisfy (4), then the polyhedron $P = P_1 \times P_2 \subset E^{\nu_1} \times E^{\nu_2}$ also satisfies (4).

3°. If a polyhedron P satisfies (4) and the point $A \overline{\in}$ aff P, then the polyhedron $P_1 = \text{conv}(A \cup P)$ also satisfies (4).

Let the set P satisfy (4).

4°. The set P is a polyhedron and all of its facade satisfies (4).

Let Q be a facade of the polyhedron P and $\dim Q = \dim P - 1$. Let us denote by T a polyhedron such that $\operatorname{extr} T = \operatorname{extr} P \setminus \operatorname{extr} Q$.

5°. The polyhedron T is situated in an affine subspace parellel to $\operatorname{aff} Q$, and is a translate of some facade of the polyhedron Q.

From the above assertions it follows that for $\dim P = 1,2,3$ a segment, triangle, parellelogram, tetrahedron, triangular prism, tetragonal pyramid with a parellelogram base and a parallelpiped, are the only figures satisfying the condition (4).

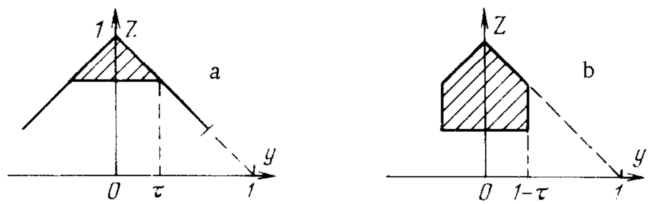

Fig. 1

Example. Let yoz denote a rectangular coordinate system, and let Eqs. (1) have the form

$$y^{\cdot} = v, \quad |v| \leqslant 1; \quad z^{\cdot} = u, \quad |u| \leqslant 1$$

The function $\varphi(y) = 1 - |y|$, and the set $G = \{y : |y| \leqslant 1\}$. The condition (5) is a Lipschitz condition with the constant equal to unity, and it holds, as well as (4), consequently according to the theorem just proved, $T_\tau^\circ(M) = T_*(M)$. The latter set is nonempty when $0 \leqslant \tau \leqslant 1$ and has the form shown in Fig. 1 for $0 \leqslant \tau \leqslant 1/2$ (a) and for $1/2 \leqslant \tau \leqslant 1$ b .

REFERENCES

1. Krasovskii, N.N., On a problem of tracking. PMM Vol. 27, No. 2, 1963.

2. Krasovskii, N.N. and Subbotin, A.I., Positional Differential Games. Moscow, " Nauka", 1974.

3. Pshenichnyi, B.N. and Sagaidak, M.I., Differential games with fixed time. Kibernetika, No. 2, 1970.

4. Gusiatnikov, P.B., On the structure of differential games. In coll. Mathematical Methods of System Investigation and Optimization. No. 3, Kiev, 1970.

5. Rockafellar, R., Convex Analysis. Princeton, N.J., Princeton University Press, 1970.

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